

《应用随机过程》第三次作业和参考答案

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问题一：

假设 $W(t), t \geq 0$ 为布朗运动, $\mathcal{F}_t, t \geq 0$ 为该布朗运动的 σ -代数流。证明 $W^2(t) - t$ 是鞅。

参考答案: $0 \leq s \leq t$,

$$\begin{aligned} & E[W^2(t) - t | \mathcal{F}(s)] \\ &= E[(W(t) - W(s))^2 + 2W(t)W(s) - W^2(s) - t | \mathcal{F}(s)] \\ &= E[(W(t) - W(s))^2 - t | \mathcal{F}(s)] + E[2W(t)W(s) | \mathcal{F}(s)] + E[-W^2(s) | \mathcal{F}(s)] \\ &= [(t - s) - t] + 2W(s)E[W(t) | \mathcal{F}(s)] - W^2(s) \\ &= W^2(s) - s \end{aligned}$$

问题二：

考虑一个在整数上的随机游动模型, 每一步的行走用随机变量 ξ_n 来表示, 设向右移动的概率为 $p < 1/2$, 向左的概率为 $1 - p$ 。即: $P(\xi_n = 1) = p$, $P(\xi_n = -1) = 1 - p$ 。记 n 步后的位置为 S_n , 即: $S_n = S_0 + \sum_{i=1}^n \xi_i$, 其中 $S_0 > 0$ 为常数, 表示初始位置。令 T 表示随机游走第一次到达 0 或者 N 的时刻, 即: $T = \inf \{n: S_n = 0 \text{ 或者 } N\}$ 。提示: σ -代数流就当作 $\mathcal{F}_n = \sigma(S_1, S_2, \dots, S_n)$ 。

- (1) 证明: $M_n = \left(\frac{1-p}{p}\right)^{S_n}$ 为鞅
- (2) 证明: T 为停时
- (3) 利用鞅的停时定理, 求 $P(S_T = 0)$ 和 $P(S_T = N)$
- (4) 证明 $Y_n := S_n + n(1 - 2p)$ 为鞅
- (5) 计算 $E(T)$

参考答案：

$$(1) \quad E[M_{n+1} | \mathcal{F}(n)]$$

$$= E \left[\left(\frac{1-p}{p} \right)^{S_{n+1}} \middle| \mathcal{F}(n) \right]$$

$$= \left(\frac{1-p}{p} \right)^{S_n} E \left[\left(\frac{1-p}{p} \right)^{\xi_{n+1}} \middle| \mathcal{F}(n) \right]$$

$$= \left(\frac{1-p}{p} \right)^{S_n} \left[\frac{1-p}{p} \times p + \left(\frac{1-p}{p} \right)^{-1} \times (1-p) \right]$$

$$= \left(\frac{1-p}{p} \right)^{S_n}$$

$$= M_n$$

So martingale.

$$(2) \quad T = \inf\{n: S_n = 0 \text{ or } S_n = N\},$$

so $\{T = k\} = \{S_i \neq 0 \& N, \text{ for all } i = 0, 1, \dots, k-1\} \cap \{S_k = 0 \text{ or } S_k = N\} \in \mathcal{F}(n)$, so T is a stopping time.

(3) Since M_n is a martingale, T is a stopping time, and:

$$a. \quad P\{T < \infty\} = 1 - P\{T = +\infty\} = 1 - \lim_{n \rightarrow \infty} \frac{1}{2^n} = 1$$

$$b. \quad E[|M_n|] \leq \left(\frac{1-p}{p} \right)^N < +\infty$$

$$c. \quad \lim_{n \rightarrow \infty} E[|M_n| \cdot I_{T \geq n}] \leq \lim_{n \rightarrow \infty} \left(\frac{1-p}{p} \right)^N \times P\{T > n\} = 0$$

Therefore, $E[M_T] = E[M_0]$

$$\Rightarrow \begin{cases} \left(\frac{1-p}{p} \right)^N P\{S_T = N\} + \left(\frac{1-p}{p} \right)^0 P\{S_T = 0\} = \left(\frac{1-p}{p} \right)^{S_0} \\ P\{S_T = N\} + P\{S_T = 0\} = 1 \end{cases}$$

$$\Rightarrow P\{S_T = N\} = \frac{\left(\frac{1-p}{p} \right)^{S_0} - 1}{\left(\frac{1-p}{p} \right)^N - 1} \quad \text{and} \quad P\{S_T = 0\} = 1 - P\{S_T = N\}$$

$$(4) \quad E[Y_n | \mathcal{F}(n-1)]$$

$$= E[\{S_n + n(1-2p) | \mathcal{F}(n-1)]$$

$$= E[S_n | \mathcal{F}(n-1)] + n(1-2p)$$

$$\begin{aligned}
&= S_{n-1} + E[\xi_n] + n(1-2p) \\
&= S_{n-1} + p - 1 + n(1-2p) \\
&= S_{n-1} + (n-1)(1-2p) \\
&= Y_{n-1}
\end{aligned}$$

(5) Since Y_n is a martingale, T is a finite stopping time, so $E[Y_T] = E[Y_0]$,

$$\begin{aligned}
&\Rightarrow E[S_T + T(1-2p)] = E[S_0] = S_0, \\
&\Rightarrow E[T] = \frac{S_0 - N \times P\{S_T = N\} - 0 \times P\{S_T = 0\}}{1-2p} \\
&= \frac{S_0 - N \times \frac{\left(\frac{1-p}{p}\right)^{S_0} - 1}{\left(\frac{1-p}{p}\right)^N - 1}}{1-2p}
\end{aligned}$$

问题三:

假设 $\xi_{n,k}, n = 1, 2, \dots, k = 1, 2, \dots$ 是独立同分布的随机变量, 有有限的二阶矩, 分别记为 $\mu = E(\xi_{n,k}), \sigma^2 = Var(\xi_{n,k})$ 。Galton-Watson branching 过程为:

$$Z_0 := 1, Z_{n+1} := \sum_{k=1}^{Z_n} \xi_{n+1,k}$$

该过程的 σ 代数流为 $\mathcal{G}_n := \sigma(Z_j: 0 \leq j \leq n), n = 0, 1, 2, \dots$

证明 $M_n := \mu^{-n} Z_n, n = 1, 2, \dots$ 是鞅。

参考答案:

$$\begin{aligned}
&E[M(n+1)|\mathcal{F}(n)] \\
&= \mu^{-(n+1)} E\left(\sum_{k=1}^{Z_n} \xi_{n+1,k} \middle| \mathcal{G}(n)\right) \\
&= \mu^{-(n+1)} \sum_{k=1}^{Z_n} E(\xi_{n+1,k} | \mathcal{G}(n)) \\
&= \mu^{-(n+1)} Z_n \mu \\
&= M_n
\end{aligned}$$

问题四：

假设 $W(t), t \geq 0$ 为布朗运动, $W(t)$ 的方差 $\sigma^2 = 9t$, 如果 $W(0) = 10$, 求 $P(W(0.5) > 10)$ 。

参考答案：

At $W(0.5)$, we have $W(0.5) = 10 + 3\sqrt{0.5}Z$, where Z is a standard normal random variable ($m = 0, s^2 = 1$). And so to calculate $P(W(0.5) > 10)$, we are calculating $P(10 + 3\sqrt{0.5}Z > 10)$. Simplifying we have:

$$P(10 + 3\sqrt{0.5}Z > 10) = P(3\sqrt{0.5}Z > 0) = 0.5$$

问题五：

假设 $B_1(t)$ 和 $B_2(t)$ 为两独立的标准布朗运动, 证明 $X_t := B_1(t) + B_2(t)$ 也为布朗运动。

参考答案：

1. Continuous path, 2. Normal increment, 3. For $t \geq s$, $X(t) - X(s) = [B_1(t) - B_1(s)] + [B_2(t) - B_2(s)]$ is stationary increment.